What does integrability of finite-gap or soliton potentials mean?

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In the example of the Schrödinger/KdV equation we treat the theory as equivalence of two concepts of Liouvillian integrability: quadrature integrability of linear differential equations with a parameter (spectral problem) and Liouville's integrability of finite-dimensional Hamiltonian systems (stationary KdV–equations). Three key objects in this field: new explicit Ψ-function, trace formula and the Jacobi problem provide a complete solution. The Θ-function language is derivable from these objects and used for ultimate representation of a solution to the inversion problem. Relations with non-integrable equations are discussed also.

Keywords: finite-gap integration; algebraic-geometric methods; soliton theory; algebraic curves; spectral problems; integrability by quadratures; Liouvillian solutions; Riccati's equation; trace formulas; Θ-functions; Abelian integrals

1. Introduction

As it is understood in contemporary language, the theory of algebraic-geometric (or finite-gap) integration is a theory of integration of integrable nonlinear (1+1)-soliton partial differential equations (PDE's), finite-dimensional Hamiltonian dynamical systems, and spectral problems defined by ordinary differential equations (ODE's). A large number of papers devoted to these problems, appeared over the last three decades, show that the vague term 'integration of integrable' does not mean an automatical integrating procedure in some simple sense of the word. The contributions that are still being made to this theory testify to its vitality: all the evidence points to the continuance of its growth.

Although the term 'completely integrable' is quite appropriate one, there is not commonly agreed answer to the questions about integrability and, in particular, to the question in the title of the present work. For example, in the theory of dynamical systems this means the well defined Liouvillian integrability, but the search for separability variables (followed by action-angle ones) is a subject of an independent theory (Flaschka & McLaughlin 1976; Vanhaecke 1996). In the case of (1+1)-integrable PDE's, the theory is, in fact, a treatment of these equations as infinite-dimensional analogues of Hamiltonian systems (Zakharov & Faddeev 1971; Gardner 1971) plus set of nontrivial exact solutions in terms of elementary functions: solitons, positons, and their relatives (Ablowitz & Segur 1981). There are natural multidimensional generalizations of the theory (the Kadomtsev–Petviashvili (KP),

Davey–Stewartson equations, their hierarchies, etc.) which are rather well developed also. Integrability of spectral problems (direct and inverse) is usually associated with the inverse scattering method, soliton theory (Ablowitz & Segur 1981; Novikov 1984; Levitan 1987) and its Θ-generalizations (Dubrovin 1975; Matveev 1976; McKean & van Moerbeke 1975; Flaschka 1975; Novikov 1984; Belokolos *et all* 1994; Gesztesy & Holden 2003). We must include here the deep links of this field to diverse areas of mathematics shedding light on the mechanism of integrability: Hirota's τ-function method, Painlevé analysis (Conte 1999), Abelian varieties (Dubrovin 1976; Vanhaecke 1996), Darboux–Bäcklund algebraic transformations (Matveev & Salle 1991), differential geometry (Darboux 1915), theory of commuting differential operators (Krichever 1978) etc. Each of these links is sufficient in itself to provide a complete development; combined, they exhibit an unusual wealth of ideas and furnish rich resources of new interrelations.

Nevertheless, there is a common object for all the approaches mentioned above: a Θ-representation for a wide class of solutions and fundamental Ψ-function as a solution to associated linear spectral problem. The key role of this object was discovered by Its & Matveev (1975) in the example of the KdV-equation. The next years after, Krichever (1976, 1977a) put such a construction into a basis of integration of soliton equations and it became clear that this link between KdV–equation and the Schrödinger operator

$$\Psi'' - u\,\Psi = \lambda\,\Psi, \qquad u = \phi(x) \tag{1.1}$$

is not an exception but a common feature. The ideology is spread to the whole class of such problems. Presently this is known as a concept of the Baker–Akhiezer function (Krichever 1978) and such an approach generates all the (1+1)-soliton equations and their hierarchies if an algebraic curve has been specified.

Originally the term 'finite-gap' meant spectral problem for the smooth real periodic potential with finitely many lacunae at spectrum of Schrödinger's operator. Later, in the 1970's, in works by Matveev, Its, Dubrovin, Krichever, and others that treatment was generalized to quasi-periodic complex valued potentials and related to methods of algebraic geometry and Θ-functions (Matveev 1976; Dubrovin *et all* 1976; Krichever 1977b). By this reason throughout the paper we keep traditional terminology 'finite-gap' but identify it with 'algebraic-geometric'.

It is difficult to keep pace with the continuing growth of the literature which is due to the activity of mathematicians. To become acquainted with the background of this field, as well as find a complete set of references, it is perhaps best to consult the surveys written by initiators of the theory.

2. Solvable potentials for the Schrödinger equation

Until the pioneer work by Novikov (1974) the Schrödinger equation (1.1) was mostly an object of the spectral theory of operators (Akhiezer 1961; Levitan & Sargsjan 1975). By that time few solvable (in different senses) examples were known.

1. The constant potential (trivial case);

2. Quantum harmonic oscillator $u = x^2$;

3. Linear potential $u = x$;

4. Decaying potentials $u = -n\,(n+1)\cosh^{-2}x$ and their rational degenerations;

5. Reflectionless potentials of Bargmann (solitons) characterized by the property $\Psi(x;\lambda) = P(x;\lambda)\exp\sqrt{\lambda}\,x$ with a polynomial in λ function P (see also Darboux (1915: p. 212));

6. The time isospectral deformations of these reflectionless potentials being governed by KdV-dynamics (Ablowitz & Segur 1981);

7. Lamé's potentials $u = n\,(n+1)\,\wp(x)$ (see Whittaker & Watson (1927));

8. Generalizations of the Lame potentials (Darboux 1882, 1915: p. 228)

$$u = \mu\,(\mu-1)\,k^2\,\frac{\mathrm{cn}^2 x}{\mathrm{dn}^2 x} + \mu'(\mu'-1)\,k^2\,\mathrm{sn}^2 x + \frac{\nu\,(\nu-1)}{\mathrm{sn}^2 x} + \nu'(\nu'-1)\,\frac{\mathrm{dn}^2 x}{\mathrm{cn}^2 x},$$

being called now the Treibich–Verdier potentials. The following year after Darboux, two comprehensive mémoires by Sparre (1883) appeared on further generalizations.

A considerable result of Matveev and Its is that they all, apart from the cases 2–3, lie within the framework of unified theory and are specifications or degenerations of a wider class, the class of algebraic-geometric potentials expressible by the following nice formula (Its & Matveev 1975a,b; Matveev 1976)

$$u = -2\,\frac{\mathrm{d}^2}{\mathrm{d}x^2}\ln\Theta\bigl(x\boldsymbol{U} + \boldsymbol{D}\bigr) + \mathrm{const.} \tag{2.1}$$

The same authors (see also Novikov 1974; Dubrovin 1975; Lax 1975) proved that all these potentials are solutions of higher stationary ODE's which are presently named Novikov's equations. An important observation of Novikov (1974) was that these equations are representable as Hamiltonian finite-dimensional dynamical systems in x. Soon this property was completely clarified by Gel'fand & Dikii (1975, 1979).

The Hamiltonian treatment of Novikov's equations closely joined the examples mentioned above with the well-known Liouville's integrability. Moreover, Liouville's integrability of these equations has received a natural completion. Namely, all the solutions are given by the famous trace formula (Matveev 1976)

$$u = 2\sum_{k=1}^{g}\gamma_k - \sum_{k=1}^{2g+1}E_k. \tag{2.2}$$

Notice that presence of Liouville's attributes itself does not automatically provide a procedure of the integration (this is a theorem of existence) but the trace formula (2.2) supplemented with the Jacobi inversion problem brings about such a procedure. In the language of dynamical systems this means, in fact, transformation to separability variables.

What could one say about solution Ψ corresponding to the potentials mentioned above? The answers are well-known. Elementary functions in the cases 1 and 4–6, Airy's special functions in the case 3, the Hermité polynomials (up to an exponential factor) for the case 2 under $\lambda = -2\,n - 1$ with integral n, Weber's (or parabolic's

cylinder) special functions for this case under arbitrary λ, and elliptic and related to them functions for the cases 7–8. For example, in the case 7 with $n = 1$ we have

$$\Psi(x; \lambda) = \frac{\sigma\big(x + \wp^{-1}(\lambda)\big)}{\sigma(x)} \mathrm{e}^{\zeta(\wp^{-1}(\lambda))\,x}. \tag{2.3}$$

The Θ-representation of the Ψ-function corresponding to the potentials (2.1), in traditional notations and terminology, is given by the formula of A. Its:

$$\Psi\big(x; \lambda(\mathcal{P})\big) = \mathrm{const}(\mathcal{P}) \frac{\Theta\big(\boldsymbol{A}(\mathcal{P}) + x\boldsymbol{U} + \boldsymbol{D}\big)}{\Theta\big(x\boldsymbol{U} + \boldsymbol{D}\big)} \mathrm{e}^{\Omega(\mathcal{P})x}. \tag{2.4}$$

On the other hand, the elementary solutions to the Ψ are representable by elementary functions, i. e. exactly solvable in terms of elementary indefinite integrals. The elliptic cases 7–8 and (2.3) are not exceptions: they are representable by indefinite elliptic integrals. A natural question arises: what about (2.1) and (2.4)? The answer is that these are not exception as well. Indefinite quadratures is a common property of the spectral problem under consideration. This means that the known Liouville's integrability of nonlinear Novikov's equations, in fact, turns out to be equivalent to the quadrature integrability of the 'linear' Ψ and conversely. In turn, the well-known effective solvability of direct/inverse spectral problems in the class of analytic nonsingular decaying soliton potentials (Ablowitz & Segur 1981) turns out to be nothing else but explicit solvability by *elementary* tools, i. e. indefinite integrations and their inversions. Moreover, such an opportunity is only one.

The next sections contain proof of the statements above and we suggest that this is a common feature of all spectral problems arising in the soliton theory. It should be emphasized here that the representation (2.4) (after crossing out λ and adding the KP-variables y, t_1, t_2, …) as a function of a point $\mathcal{P}$ on arbitrary algebraic curve is a natural and fundamental object for KP-hierarchies of $(2+1)$-PDE's (Krichever 1977). We shall restrict our consideration only to spectral problems defined by ODE's as independent objects, so that their integrability, in the above sense, belongs to their intrinsic nature. Moreover, we will restrict ourselves only to the Schrödinger equation (1.1) that we view merely as a differential equation with a parameter (subject to explicit integration) rather than a spectral problem with any boundary conditions, commuting operator, etc.

3. Jules Drach

Perhaps the most surprising facet is the fact that the ideology mentioned above has not received mention in the modern literature in the context. It belongs to J. Drach (1919) and his name was revealed by D. & G. Chudnovsky (1984) and V. Matveev (Belokolos *et all* 1994: pp. 84–85), who drew attention his remarkable results. The first sentences in (Drach 1919) clearly indicate his motivations†.

† …where h is an *arbitrary parameter*. The most interesting among them are those where the Riccati equation

$$\rho' + \rho^2 = \varphi + h$$

(and consequently the equation $\frac{d^2y}{dx^2} = [\varphi(x) + h]\, y$ as well) can be integrated by *quadratures*; we will show how one determines the function φ in all these cases.

Theorem 3.1 (Drach's Thesis). *The class of finite-gap* (*algebraic-geometric*) *potentials is the only one when the spectral problem* (1.1) *is integrable for the* Ψ *by quadratures under all values of the parameter* λ. *The solution has the form*

$$\Psi_\pm(x;\lambda) = \sqrt{R(x;\lambda)} \; \exp\!\int^x \frac{\pm\mu \, \mathrm{d}x}{R(x;\lambda)}, \qquad (3.1)$$

wherein the function R *is a polynomial in* λ *solution of the equation*

$$\mu^2 = -\frac{1}{2}\,R\,R'' + \frac{1}{4}R'^2 + (u+\lambda)\,R^2. \qquad (3.2)$$

Drach himself does not give a proof therefore to produce one, including step-by-step mechanism of integration, is not without interest. The proof will call to mind the classical result of Its & Matveev (1975) about criteria of the potential to be finite-gap. However we do not invoke any attributes of spectral theories: reality of the potential, periodicity, Weyl's bases, monodromies, squares of eigenfunctions $R = a\Psi_1^2 + b\Psi_1\Psi_2 + c\Psi_2^2$, spectrums, resolvent, functional spaces, etc.

Actually we will be writing known and somewhat new formulas of the theory but keeping in mind only ideology of Drach. Before passing to the proof of the theorem we will formulate a known statement which occurs in the literature in numerous contexts (Ermakov 1880; Marchenko 1974; Its & Matveev 1975; Gel'fand & Dikii 1975; Al'ber 1979).

Proposition 3.2. *Integrability of the equation* (1.1) *is equivalent to integrability of the third order linear differential equation*

$$R''' - 4\,(u+\lambda)\,R' - 2\,u'R = 0 \qquad (3.3)$$

or compact nonlinear equation of Ermakov for quasi-amplitude $\Xi = \sqrt{R}$ *of the* Ψ:

$$\Xi'' - (u+\lambda)\,\Xi = -\frac{\mu^2}{\Xi^3}. \qquad (3.4)$$

Though this proposition is a straight consequence of the equations (3.1–2), we will give a *derivation* all the formulas (3.1–4). Arguments are as follows.

One sufficient test of quadrature integrability a given ODE is furnished by a solvable Lie point symmetry of the equation. This technique (group analysis of differential equations) is rather well developed (Ibragimov 1985; Eisenhart 1933) so that applying this simple computations to the Schrödinger equation (1.1) we get necessary attributes of the theory in a natural way. Indeed, generator $\widehat{\mathfrak{G}}$ of the point symmetries $(x,\,\Psi) \overset{\varepsilon}{\mapsto} (\widetilde{x},\,\widetilde{\Psi})$ for the equation (1.1) has the following form:

$$\widehat{\mathfrak{G}} = \left(a\,\Psi + 2\,R\right)\partial_x + \left\{a_x\Psi^2 + (R' + \text{const.})\,\Psi + b\right\}\partial_\Psi, \qquad (3.5)$$

wherein $a = a(x;\lambda)$, $b = b(x;\lambda)$ are arbitrary solutions of (1.1) and $R = R(x;\lambda)$ satisfies the equation (3.3). The functions $a(x;\lambda)$, $b(x;\lambda)$ do not help in further integrating (need to know solution Ψ itself) and we set them equal to zero. The remaining free constant in (3.5) says that the symmetry does not disappear and we have a solvable commutative 2-parametric symmetry $\big\langle \widehat{\mathfrak{G}}_1 = \Psi\,\partial_\Psi, \; \widehat{\mathfrak{G}}_2 = 2R\,\partial_x +$

$R'\Psi\,\partial_\Psi\rangle$. Transformation to 'integrable' variables $(x,\Psi)\mapsto(z,w)$ is provided by the standard Lie symmetry machinery. We get an explicit change (Eisenhart 1933: p. 91, case 3°)

$$z=\int^x\frac{\mathrm{d}x}{R(x;\lambda)},\qquad w=\ln\frac{\Psi}{\sqrt{R(x;\lambda)}}.$$

In new variables, as Lie's theory guarantees, the equation will be easily solvable. Making use of one-fold integrated form of the equation (3.3), i.e. the equations (3.2) or (3.4), an equation for the function $w=w(z)$ becomes $w_{zz}+w_z^2=\mu^2$, where the constant of Ermakov–Drach μ is a constant of integration. The equation is readily solved indeed and, after back transformations, we arrive at the formula (3.1). Point of departure of Ermakov (1880) was the same—integrability by quadratures.

4. Integrability of the Schrödinger equation: Theorem 3.1

(a) Proof

Necessity. The function Ψ followed by the R is a function of two variables x and λ. Our aim is to find out the dependence of these functions upon both the variables. They are analytical entire functions of the parameter λ (Yakubovich & Starzhinskii 1975). This allows one to represent R in a form of analytical series

$$R(x;\lambda)=R_0(x)+R_1(x)\,\lambda+R_2(x)\,\lambda^2+\cdots+R_k(x)\,\lambda^k+\cdots. \qquad (4.1)$$

The equation (3.3) is linear hence there exists a recurrence relation on the coefficients R_k derivable from the equation (3.3)

$$R_{k-1}=\widehat{\mathcal{K}}\,R_k,\qquad \widehat{\mathcal{K}}\equiv\frac{1}{4}\,\partial_{xx}-u+\frac{1}{2}\int^x u_x\ldots\,\mathrm{d}x\ . \qquad (4.2)$$

The next step is to make use of Ermakov–Drach's equation (3.2) wherein the constant μ is, at the moment, *arbitrary and independent of* λ. We must require integrability under *all* values λ. This means that after substitution the series (4.1) into (3.2), coefficients in front of λ^k must be zeroes, independently of each other:

$$\left(\mu^2+\frac{1}{2}R_0R_0''-\frac{1}{4}R_0'^{\,2}-uR_0^2\right)+\left(\frac{1}{2}(R_0R_1)''-\frac{3}{2}R_0'R_1'-2\,uR_0R_1-R_0^2\right)\lambda+\cdots.$$

Hence, the coefficients $R_k(x)$ are determined by subsequent integration this infinite system of chained equations. It is clear that quadrature integrability takes place not for arbitrary functions $u=\phi(x)$. By which restrictions are such functions distinguished from all possible ones?

Rewrite the series (4.1) as a power series in $\zeta=\lambda^{-1}$ (Gel'fand & Dikii 1975):

$$R(x;\lambda)=\widetilde{R}_0(x)+\widetilde{R}_1(x)\,\zeta+\widetilde{R}_2(x)\,\zeta^2+\cdots+\widetilde{R}_k(x)\,\zeta^k+\cdots,$$

whereupon the recurrence relation (4.2) acquires the well-known computable form:

$$\widetilde{R}_k=\widehat{\mathcal{K}}\,\widetilde{R}_{k-1}:\quad R(x;\lambda)=1-\left(\frac{1}{2}\,u-c_1\right)\zeta-\left(\frac{1}{8}\,u_{xx}-\frac{3}{8}\,u^2+\frac{1}{2}\,c_1u-c_2\right)\zeta^2+\cdots.$$

Thus, for arbitrary $u = \phi(x)$ the Ψ is expressed by the formula (3.1) with the differential polynomial $R([u]; \lambda)$ of infinite order. Moreover, making use of this recurrence with subsequent collection in ζ, the equation (3.2) turns into

$$\zeta \, \mu^2 = 1 + 2 \, c_1 \zeta + \left(2 \, c_2 + c_1^2\right)\zeta^2 + 2 \left(c_3 + c_1 c_2\right)\zeta^3 + \cdots + \text{second half} \,. \qquad (4.3)$$

The dependence on u has gone at infinity: the first half (it is infinite!) contains only constants c_k and the second one does differential polynomials of infinite order in u. In order to get conditions of finite order in derivatives $u^{(k)}$ we have the only possibility—to set R to be a finite polynomial:

$$R([u]; \lambda) = \lambda^g + R_{g-1}\lambda^{g-1} + \cdots + R_1\lambda + R_0. \qquad (4.4)$$

Accordingly, splitted restrictions on the potential (4.3) become g, now not necessary zero, constants $I_k = F_k(c_1, c_2, \ldots, c_g)$ plus $g+1$ differential conditions of finite order

$$I_k = F_k\big(u, \, u_x, \ldots, \, u^{(2g)}; \, c_1, c_2, \ldots, c_g\big), \qquad k = g + 1 \ldots 2\,g + 1. \qquad (4.5)$$

Their compatibility means that such a class of potentials is not empty and the parameter μ *must depend on the spectral one*. This corresponds to a choice the particular solution R among three linear independent $R_{1,2,3}$ defined by integration constants $A_{1,2,3}$ such that the constant $\mu(A_{1,2,3})$ is algebraically related with λ: $\mu = \mu(\lambda)$. These potentials admits such a dependence and the condition (4.3), or which is the same (3.2), turns into a hyperelliptic algebraic curve of finite genus

$$\mathfrak{S}: \qquad \begin{aligned} \mu^2 &= \lambda^{2g+1} + I_1\lambda^{2g} + \cdots + I_{2g}\lambda + I_{2g+1} = \\ &= (\lambda - E_1)\cdots(\lambda - E_{2g+1}). \end{aligned} \qquad (4.6)$$

The set of equations (4.5) is 1-fold integrated Novikov's equations (stationary KdV-equations) and a half of their integrals $I_k = F_k([u]; c)$. Thus quadrature integrability of the Ψ has became equivalent to integrability of these equations. If it has been established, a final answer is given by substitution $u = \phi(x)$ into the polynomial (4.4) and then into the formula (3.1). We emphasize that the well-known formula (3.1) *does not mean any integrability* without formula for $\phi(x)$. This is ansatz and such a form of solution to the ODE (1.1) does exist for arbitrary $\phi(x)$.

Sufficiency. The sufficiency is a procedure of explicit integration of the equations (4.5). It is widely known and we do not repeat details here (Drach 1919; Dubrovin 1975). The answer is the formula (2.2) plus integral representation for the functions $\gamma_k(x)$. Nevertheless we note that being a remarkable *identity* in the spectral theories (Levitan 1987), the trace formula (2.2) turns into a necessary and key object in the quadrature methodology†. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\Box$

(b) *Drach–Dubrovin equations and formula for the Ψ-function*

Rather than mere identities, we consider consequences of the well-known definition of the polynomial R and new variables γ_k, for example $u_{xx} - 3\,u^2 \cong \sum \gamma_j\gamma_k$, as algebraic transformations from the variables $\{u, u_x, \ldots\}$ to the separability variables

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† Independent role of these formulas for the KdV–equation, as well as their dynamical relatives, was pointed out by Matveev (1975). See also his appendix to (Dubrovin *et all* 1976).

$\{\gamma, \mu\}$. Namely, writing the function

$$R([u]; \lambda) = \lambda^g - \left(\frac{u}{2} - c_1\right)\lambda^{g-1} - \left(\frac{1}{8}u_{xx} - \frac{3}{8}u^2 + \frac{1}{2}c_1 u - c_2\right)\lambda^{g-2} + \cdots \quad (4.7)$$

in factorized form $R([u]; \lambda) = \big(\lambda - \gamma_1(x)\big) \cdots \big(\lambda - \gamma_g(x)\big)$, we *do define* the first half of the change of variables as *zeroes* of this R:

$$\{u, u_x, \ldots, u^{(2g-1)}\} \mapsto \{(\gamma_1, \gamma_2, \ldots, \gamma_g), \quad (\mu_1, \mu_2, \ldots, \mu_g)\}. \quad (4.8)$$

More precisely, the first part of the change (4.8) is as follows

$$\begin{cases} u &= 2\sum_{k=1}^{g}\gamma_k + 2c_1, \quad 2c_1 = -\sum_{k=1}^{2g+1}E_k \\[2mm] u_{xx} &= 12\sum_{k=1}^{g}\gamma_k^2 - 8\sum_{k,\,j>k}^{g}\gamma_k\gamma_j - 16c_1\sum_{k=1}^{g}\gamma_k + 4c_1^2 + 8c_2 \\[2mm] \ldots\ldots & \end{cases}$$

and not closed due to the odd derivatives of u. Missing ones and therefore the second half is extracted by involving the Drach–Dubrovin differential equations:

$$\frac{\mathrm{d}\gamma_k}{\mathrm{d}x} = \frac{-2\mu_k}{\prod\limits_{j \neq k}(\gamma_k - \gamma_j)}, \qquad \mu_k^2(x) = \big(\gamma_k(x) - E_1\big) \cdots \big(\gamma_k(x) - E_{2g+1}\big), \quad (4.9)$$

where constants E_j are functions of Novikov's constants c_j and the integrals (4.5). We thus get the remaining part of the complete change (4.8):

$$\begin{cases} u_x &= -4\sum_{k=1}^{g}\frac{\mu_k}{\prod\limits_{j\neq k}(\gamma_k - \gamma_j)} = \left(2\sum_{k=1}^{g}\gamma_k + 2c_1\right)_x \\[4mm] u_{xxx} &= 16\sum_{k=1}^{g}\frac{(2c_1 - 3\gamma_k)}{\prod\limits_{j\neq k}(\gamma_k - \gamma_j)}\mu_k + 16\sum_{k,\,j>k}^{g}\left\{\frac{\mu_k\gamma_j}{\prod\limits_{n\neq k}(\gamma_k - \gamma_n)} + \frac{\mu_j\gamma_k}{\prod\limits_{n\neq j}(\gamma_j - \gamma_n)}\right\} \\[4mm] \ldots\ldots & \end{cases}.$$

The equations (4.9) are readily rewritten into a promised integral form which coincides the Jacobi inversion problem for the hyperelliptic algebraic curve (4.6)

$$\begin{cases} \displaystyle\int\limits^{(\gamma_1,\mu_1)}\frac{\mathrm{d}z}{w} + \cdots + \displaystyle\int\limits^{(\gamma_g,\mu_g)}\frac{\mathrm{d}z}{w} &= d_1 \\[4mm] \displaystyle\int\limits^{(\gamma_1,\mu_1)}z\,\frac{\mathrm{d}z}{w} + \cdots + \displaystyle\int\limits^{(\gamma_g,\mu_g)}z\,\frac{\mathrm{d}z}{w} &= d_2 \\[2mm] \ldots\ldots \quad \ldots \quad \ldots\ldots \quad \ldots \\[2mm] \displaystyle\int\limits^{(\gamma_1,\mu_1)}z^{g-1}\,\frac{\mathrm{d}z}{w} + \cdots + \displaystyle\int\limits^{(\gamma_g,\mu_g)}z^{g-1}\,\frac{\mathrm{d}z}{w} &= d_g - 2x \end{cases}. \quad (4.10)$$

With this equation we can obtain formula for the Ψ-function (3.1) which is not seem to be in the literature:

$$\Psi_\pm(x;\lambda) = \exp\frac{1}{2}\left\{ \int^{\gamma_1(x)} \frac{w\pm\mu}{(z-\lambda)\,w}\,\mathrm{d}z \;+\cdots+\; \int^{\gamma_g(x)} \frac{w\pm\mu}{(z-\lambda)\,w}\,\mathrm{d}z \right\}. \qquad (4.11)$$

The variables of integration lie on the curve $w^2 = (z-E_1)\cdots(z-E_{2g+1})$.

It may be remarked at once that complete set of the transformations above to the separability variables $\{\gamma,\mu\}$ is not necessary for Novikov's equations themselves. This is an attribute of their Hamiltonian description (Gel'fand & Dikii 1979) derivable from the main trace formula (2.2) and the equations (4.9). It thus appears that the formula (3.1) or (4.11) supplemented with the two objects (2.2) and (4.9–10) explains the nature of integrability in question. Subsequent procedure of inversion for a symmetrical sum of γ's is necessary for ultimate representation to the solution. In the degenerated cases (constant potential, solitons and the like) all the integrals (4.10) reduce to integrals of rational functions, so that the problem becomes trivial (inversion of logarithms) and leads to well-known exponents. In general case, this transcendental problem is solved with the help of Riemann's Θ-functions (see § 7).

We would like to note that even 'naive' (not finite-gap) procedure of integration of the trivial potential $\Psi'' - \mathrm{const.}\Psi = \lambda\Psi$ completely accords with the scheme described above. Holomorphic integrals do not appear but, instead, the following objects arise: integral definition of the logarithm, i.e. integration of a trivial rational function, and necessity of inversion of the former (the exponent).

5. Some consequences

(a) Integrable λ-pencils

Quadrature integrability does not depend on a choice of dependent/independent variables. We could formally avoid the inversion of Abelian integrals (4.10) rewriting the theory in the 'inverse' variable u. Indeed, after the change $(x,\Psi)\leftrightarrow(u,Y)$

$$x = \chi(u), \qquad \Psi = \sqrt{\chi_u}\,Y \qquad (5.1)$$

we arrive at a second order linear ODE ($\{\chi,u\}$ is the standard schwarzian)

$$Y_{uu} = Q(u;\lambda)\,Y, \qquad Q(u;\lambda) = -\frac{1}{2}\{\chi,u\} + (u+\lambda)\,\chi_u^2 \qquad (5.2)$$

which can be of interest in its own right as a new spectral problem (operator λ-pencil) for some 'good' functions $Q(u;\lambda)$. The former depends on a chosen potential $x=\chi(u)$. Apparently the second equation in (5.2), written in the form

$$F\,F_{uu} - \frac{3}{2}\,F_u^2 - 2\,(u+\lambda)\,F^4 + 2\,Q(u)\,F^2 = 0, \qquad F\equiv\chi_u \qquad (5.3)$$

and equation (6.4) play an important part in the theory†, if only because inversion of arbitrary finite-gap potential satisfies this integrable 2-nd order nonlinear ODE with suitable function Q. Let us consider some examples $\big(N = n\,(n+1)\big)$.

† See also a footnote on p. 97 in (Gel'fand & Dikii 1975) apart from two misprints in one term.

- Soliton potentials $u = -n\,(n+1)\cosh^{-2}x$:

$$Q(u;\lambda) = -\frac{3}{16}\frac{1}{(u+N)^2} + \frac{\lambda-1}{4\,u^2} - \frac{\lambda-N-1}{4\,u\,(u+N)};$$

- The Lamé potentials $u = n\,(n+1)\,\wp(x;g_2,g_3)$ (Whittaker & Watson 1927):

$$Q(u;\lambda) = -\frac{3}{16}\sum_{k=1}^{3}\frac{1}{(u-Ne_k)^2} + \frac{1}{8}\frac{(3+2\,N)\,u+2\,N\,\lambda}{(u-Ne_1)(u-Ne_2)(u-Ne_3)};$$

- Arbitrary even elliptic finite-gap potential $u = U\big(\wp(x)\big)$, where U is a rational function of $\wp$. A wide family of such potentials there provides the theory of elliptic solitons (Acta Appl. Math. 1994). The function Q is as follows

$$Q = \frac{1}{2}\frac{\{U,\,\wp\}}{U_\wp^2} + \frac{U\big(\wp(x)\big)-3\,\wp(2x)+\lambda}{U_\wp^2\,\wp_x^2} = \widetilde{U}\big(\wp(x);\lambda\big),$$

  where $\widetilde{U}$ is another rational function of $\wp$. Hence $Q = \widetilde{U}\big(U^{-1}(u);\lambda\big)$ is a genus zero algebraic function of u or rational function of $\wp$: $Q = \widetilde{Q}(\wp;\lambda)$;

- Arbitrary elliptic soliton $u = \phi(x)$. The equation (5.2) takes the form

$$Y_{uu} = \left\{\frac{\Phi_{uv}\,\Phi_u}{v\,\Phi_v^2} - \frac{1}{2}\frac{\Phi_{uu}\,\Phi_v^2+\Phi_{vv}\,\Phi_u^2}{v\,\Phi_v^3} - \frac{1}{4}\frac{\Phi_u^2}{v^2\,\Phi_v^2} + \frac{u+\lambda}{v^2}\right\}Y,$$

  where $\Phi(u,v) = 0$ denotes a differential equation connecting the elliptic function u and its derivative $u_x = v$ (algebraic equation of genus unity).

By the previous constructions, all these equations and their relatives of the type (3.3) are of Fuchsian class and integrable by quadratures. We thus get a solution for all elliptic solitons generalizing formulas of Hermite (Belokolos *et all* 1994: p. 82):

$$Y(u;\lambda) = \sqrt{\mathcal{R}(u,v;\lambda)}\,\exp\int^{u}\frac{\mu\,\mathrm{d}u}{\mathcal{R}(u,v;\lambda)}, \tag{5.4}$$

wherein, owing to (4.7), $\mathcal{R}(u,v;\lambda) = v\,R([u];\lambda)$ becomes rational function in (u,v). The last step is a standard problem to representation the elliptic Abelian integral (5.4), which is solved in terms of Jacobian θ-functions.

It should be remarked that the presence of explicit Ψ leads to explicit factorization all of the linear operators (1.1), (3.3) and (5.2) with arbitrary λ's. For example:

$$\partial_{xx} - (u+\lambda) = (\partial_x+p)(\partial_x-p) = 0,$$

where roots $\pm p(x;\lambda)$ have the quadrature form

$$p(x;\lambda) = \frac{\frac{1}{2}R'+\mu}{R} = \mu\prod_{k=1}^{g}(\lambda-\gamma_k)^{-1} - \frac{1}{2}\sum_{k=1}^{g}\frac{\gamma_k'}{\lambda-\gamma_k}.$$

The Θ-functional representation for this and other factorizations is readily written down using formulas of §7.

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(*b*) *Liouvillian integrabilities*

To all appearances Liouville was the first (1833–41) to recognize a significance of the object $R = \Psi_1 \Psi_2$ and associated linear ODE of higher order in the context of integration of linear ODE's in closed form (Liouville 1839). The third order equation (3.3) explicitly arose and was discussed on pp. 430–431 in (Liouville 1839). Though Liouville was doing in the spirit of algebraic solvability†, a presence of a parameter in equations turns the theory into the spectral one. The polynomials in λ arose also in works by Darboux and polynomial in $\wp(x)$ was considered by Hermité in the case of Lamé's potentials $u = n(n+1)\wp(x)$ (Whittaker & Watson 1927). This corresponds exactly to the 'polynomial in u' cases of Liouville (1839) for the equation (3.3).

V. Kuznetsov (2001, personal communication) pointed out a relationship of the theory with an algorithm of Kovacic (1986) and, as we have seen now, this key observation leads to the natural conclusion:

- *Liouvillian integrability of linear* ODE*'s* (1830–40's) *with a parameter in finite terms is equivalent to algebraic integrability by Liouville* (1840–50's) *of nonlinear Hamiltonian systems*†.

The theory and examples above show that this is not a coincidence and, probably, the modern efficient computer-algorithmic theory, being applied to both equations (1.1) and (3.3), would provide the independent approaches to generation/classification of integrable linear operator pencils. For excellent explanation of Liouville's ideas see books by Mordukhai-Boltovskoi (1910), Ritt (1948), references in works by the authors mentioned in the last footnote and the Chapter IX in (Lützen 1990). Among other things this book provides a full account of references for further study. The next section contains an additional information, examples, and connections with non-integrable equations.

6. Integrable cases of Riccati and related equations

For a simplicity and to avoid lengthening the terminology we will refer to the second order linear ODE's (or potential) and corresponding to them Riccati's equations of general form $y_z + a(z) y + b(z) y^2 = c(z)$ as one object. Well-known transformations between them have a quadrature characterization.

(*a*) *Riccati's equations*

The Q-functions corresponding to integrable Riccati's equations (5.2) can be rational/algebraic, elementary, or transcendental. It is rather evident that non-integrable equations contain integrable subcases. Say, the quantum harmonic oscillator $Y_{uu} =$

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† See however p. 456 in (Liouville 1839) about what is nowadays named Liouville's extension.

† We speak here only about general link and avoid discussion the rigorous correspondence between finite-gap operators and Liouvillian solutions, differential algebra (Ritt 1948), algorithms of Singer (1981), Kovacic (1986), Picard–Vessiot theory (van der Put & Singer 2003), etc. In particular, we do not touch an important question: when does isomorphism between these two Liouvillian integrabilities take place? We should mention here some comments about this analogy in (Morales-Ruiz 1999: pp. 51–52) (see also van der Put & Singer 2003). However main attributes of the theory (spectral curves, polynomial in λ, Θ-functions, etc.) are not discussed in these works.

$(u^2 - 2\,\mu - 1)Y$ shows that this potential, seemingly having nothing in common with the finite-gap ones, is integrable by quadratures if μ is an integer.

Let us reverse a view on the equations and change (5.1–2). Whether exists transformation between nonfinite-gap equation (with a parameter or no)

$$Y_{zz} = Q(z)\,Y \tag{6.1}$$

and a finite-gap one? Such a transformation depends on chosen equations and can be rather complicated. In contrast to the preceding (5.1–2), corresponding functional relation is given by the general change of variables $(x, \Psi) \mapsto (z, Y)$

$$x = \chi(z), \qquad \Psi = \sqrt{\chi_z}\, Y \tag{6.2}$$

and depends on the potential $u = \phi(\chi)$.

Proposition 6.1. *Arbitrary equation* (6.1) *and the finite-gap one* (1.1) *are transformable into each other by the following functional relation*

$$\chi: \quad \frac{\Psi_1(x; \lambda)}{\Psi_2(x; \lambda)} = \frac{Y_1(z)}{Y_2(z)}, \tag{6.3}$$

where functions $\Psi_{1,2}(x; \lambda)$, $Y_{1,2}(z)$ *are independent solutions of* (1.1) *and* (6.1).

Proof. From (1.1) and (6.1–2) we have

$$-\frac{1}{2}\{\chi,\, z\} + \big(\phi(\chi) + \lambda\big)\,\chi_z^2 = Q(z). \tag{6.4}$$

Clearly, the sought for functional relation is an integral of this equation. The potential $\phi(\chi)$ is defined by the corresponding Ψ which is known. From the second equality in (6.2) we have

$$\frac{\mathrm{d}\chi}{\Psi^2} = \frac{\mathrm{d}z}{Y^2}.$$

Integrating and supplementing with the property

$$\int \frac{\mathrm{d}x}{\Psi_1^2} = \frac{\Psi_2}{\Psi_1}$$

we get the formula (6.3) and complete the proof. $\qquad\square$

It thus appears that the product solution $R = \Psi_1 \Psi_2$ is a fundamental object in the finite-gap theory and the ratio (6.3) is fundamental in transformations between Riccati's equations. Such arguments might seem to be trivial because every integrable equation is transformable to the trivial $Y_{zz} = 0$. But an example of Rawson (1883) shows nontrivial consequences: generating of finite-gap spectral problem (1.1) with $\phi(x) = n\,(n+1)\,x^{-2}$ from the oldest and classical equation of Riccati–Bernoulli with a parameter. The paper of Rawson is so short that we completely reproduced it in the Appendix A without any comments[†].

The following extra examples exhibit a functional relation between the Lamé potentials and equidistance spectrum of harmonic oscillator $Y_{zz} = \big(z^2 - 2\,\mu - 1\big)Y$.

[†] See also Liouville's (1841: pp. 11–13) classical considerations on integrability of the potential $\phi(x) = B\,x^{-2}$ with appearance $B = n(n+1)$. Another example of 'triviality' is the class of potentials in elementary functions (solitons and the like) generated from the zero potential by the Darboux transformation (Matveev & Salle 1991).

- $\phi(x) = n\,(n+1)\,x^{-2}$ and $\mu = 0$. We obtain

$$\chi: \quad x^{2n+1} = \int^{z} \mathrm{e}^{z^2}\,\mathrm{d}z, \qquad \lambda = 0;$$

- The Lamé 1-gap potential $\phi(x) = 2\,\wp(x)$. The relation (6.3) has the form

$$\chi: \quad \frac{\sigma(\alpha - x)}{\sigma(\alpha + x)}\,\mathrm{e}^{2\zeta(\alpha)x} = \int^{z} \frac{\mathrm{e}^{z^2}}{H_{\mu}^2(z)}\,\mathrm{d}z, \qquad \alpha = \wp^{-1}(\lambda).$$

As we have seen, these formulas are representable by quadratures in both variables. List of examples along these lines may be readily extended. For example one to derive a generalization of Rawson's transformation (see Appendix A) from the classical equation of Riccati to the Lame one. A counterexample of Liouville–Airy $Y_{zz} = (z + \alpha)Y$ brings out the differential field independently of the parameter α:

$$\chi: \quad \exp\int^{x} \frac{2\,\mu\,\mathrm{d}x}{R([u];\lambda)} = \int^{z} \frac{\mathrm{d}z}{\mathrm{Ai}^2(z + \alpha)},$$

as might be expected for the variable z.

In a broad sense, the variables $u = \phi(x)$ and $\Psi(x;\lambda)$ may be thought of as 'convenient' variables for all integrable Riccati's equations with a nontrivial parameter because, in this case, the differential polynomial $R([u];\lambda)$ has a universal description as $u = \phi(x)$ is a solution of the equations of Novikov.

(b) Related equations

The equations of Riccati, Ermakov–Drach, Novikov's equations, and equations of the type (5.1–3) and (6.4) are hidden forms of integrability of one another but are not only integrable linear/nonlinear equations arising in this theory. Besides them, the Ψ-function itself satisfies some nonlinear homogeneous autonomous differential equation of the third order with a parameter(s). That equation and its relatives are obtained with the help of suitable elimination of the potential u. The following instances illustrate the remark above.

- $\phi(x) = n\,(n+1)\,\wp(x+c)$:

$$n\,(n+1)\big(\Psi\,\Psi''' - \Psi'\Psi''\big)^2 = 4\,\Psi\big(\Psi'' - \lambda_1\Psi\big)\big(\Psi'' - \lambda_2\Psi\big)\big(\Psi'' - \lambda_3\Psi\big),$$

  where arbitrary parameters λ_k are restricted by the relation $\lambda_1 + \lambda_2 + \lambda_3 = 0$.

- Arbitrary elliptic finite-gap potential u. Then the Ψ satisfies the equation

$$\Phi\Big(\frac{\Psi''}{\Psi} - \lambda,\ \frac{\Psi'''}{\Psi} - \frac{\Psi'\Psi''}{\Psi^2}\Big) = 0,$$

  where $\Phi(u,v) = 0$ is an algebraic relation between u and its derivative $u_x = v$.

The result of elimination depends on a chosen independent variable x, u, ... and all these integrable equations and their t-deformations can be of interest in their own right if only because they are closely related to known 3-rd order autonomous nonlinear ODE of Jacobi for the ϑ-constants in the framework of Fuchsian equations. We do not develop this topic here. It would appear reasonable that the widely known and universal Θ-description of the theory (Krichever 1977a,b) should be obtainable from the spectral problem itself. This is so indeed.

7. Θ

Here we will obtain the Θ-representation (2.4) for the Ψ-function and, thereby, its properties as a function of Baker–Akhiezer. Insomuch as regular derivation of this axiomatic representation is not described in the literature, one to write up that procedure is, perhaps, not without interest.

As we mentioned above, pure spectral approaches were extended to the complex valued potentials. (See recent monograph by Gesztesy & Holden (2003) for most exhaustive bibliography and new results in spectral treatment of the theory). Taking this into account we will refer a general λ-dependence of the Ψ as its spectral property and its dependence upon x-variable is considered to be parametric. Such a spectral view arose in the paper by Akhiezer (1961) and was completed in full by Its & Matveev (1975: §4). It seems helpful to compare this classical approach with a 'reverse' one, i. e. primary x-dependence of the Ψ. This would correspond to pure quadrature arguments with a parametrical λ-dependence made out in the previous sections.

Theorem 7.1. *The Θ-functional representation* (2.4) *to the Ψ-function is a consequence of the quadrature representations* (3.1) *and* (4.11).

Proof. Since the spectral parameter λ is connected with the variable μ by the algebraic equation (4.6) of finite genus g we view both these variables as meromorphic functions $\lambda = \lambda(\tau)$, $\mu = \mu(\tau)$ of a global parameter τ on the curve (4.6). Accordingly, we consider the Ψ-function (4.11) as a function of x and τ

$$\Psi(x;\tau) = \mathrm{e}^{\dfrac{1}{2}\sum\limits_{k=1}^{g}\int\limits_{\alpha_k}^{\gamma_k(x)}\dfrac{w+\mu(\tau)}{(z-\lambda(\tau))w}\,\mathrm{d}z}, \tag{7.1}$$

where α_k are arbitrary constants. It is a symmetrical function of the quantities γ_k and the formers, as functions of x, are defined from the inversion problem (4.10):

$$\begin{cases} \displaystyle\int\limits_{\alpha_1}^{\gamma_1}\mathrm{d}\omega_1(z) + \cdots + \int\limits_{\alpha_g}^{\gamma_g}\mathrm{d}\omega_1(z) = d_1 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \displaystyle\int\limits_{\alpha_1}^{\gamma_1}\mathrm{d}\omega_g(z) + \cdots + \int\limits_{\alpha_g}^{\gamma_g}\mathrm{d}\omega_g(z) = d_g - 2\,x. \end{cases} \tag{7.2}$$

Let A_{jk} be a matrix of $\boldsymbol{a}$-periods of the holomorphic Abelian integrals (7.2):

$$\oint_{a_k} \mathrm{d}\omega_j(z) = A_{jk}.$$

All the terminology and notation in this section is standard and elucidated in any paper on the finite-gap integration. As usual, we normalize the integrals $\omega_j(z)$, introducing the canonical base of normal holomorphic Abelian integrals $\widetilde{\boldsymbol{\omega}}(z)$:

$$\omega_j(z) = A_{jk}\,\widetilde{\omega}_k(z), \qquad \oint_{a_k} \mathrm{d}\widetilde{\omega}_j(z) = \delta_{jk}, \qquad \oint_{b_k} \mathrm{d}\widetilde{\omega}_j(z) = B_{jk}. \qquad (7.3)$$

Jacobi's inversion problem (4.10), (7.2) acquires the form

$$\sum_{k=1}^{g} \widetilde{\omega}_j^{\gamma_k,\alpha_k} = -x\,U_j + C_j, \qquad U_j = 2\big(A^{-1}\big)_{jg}, \qquad (7.4)$$

where we adopt the concise Baker's notation (Baker 1897) for Abelian integrals:

$$\widetilde{\omega}_j^{\gamma_k,\alpha_k} \equiv \int_{\alpha_k}^{\gamma_k} \mathrm{d}\widetilde{\omega}_j(z) = \widetilde{\omega}_j(\gamma_k) - \widetilde{\omega}_j(\alpha_k).$$

The vector $\boldsymbol{U}$ depends on the curve $\boldsymbol{\mathfrak{S}}$ but α_k and C_j are arbitrary constants. The function (7.1) depends on the point τ on the curve $\boldsymbol{\mathfrak{S}}$ and on the variable x through γ's. Thus it is a single-valued function (but not Abelian) of a point $\boldsymbol{\eta}(x) = -x\boldsymbol{U} + \boldsymbol{C}$ on the Jacobian $Jac(\boldsymbol{\mathfrak{S}})$ and, hence, has a Θ-functional representation. Indeed, the arisen sum (7.1) is nothing else but the fundamental T-function of Weierstrass (1856) and Clebsch–Gordan (Clebsch & Gordan 1866)

$$T_{\eta\xi}\!\begin{pmatrix} \gamma_1,\ \gamma_2,\ \ldots,\ \gamma_g \\ \alpha_1,\ \alpha_2,\ \ldots,\ \alpha_g \end{pmatrix} = \int_{\alpha_1}^{\gamma_1} \mathrm{d}\widetilde{\Pi}_{\xi\eta}(z) + \int_{\alpha_2}^{\gamma_2} \mathrm{d}\widetilde{\Pi}_{\xi\eta}(z) + \cdots + \int_{\alpha_g}^{\gamma_g} \mathrm{d}\widetilde{\Pi}_{\xi\eta}(z),$$

where $\mathrm{d}\widetilde{\Pi}_{\xi\eta}(z)$ denotes the elementary normal Abelian differential of 3-rd kind with 1-st order poles at the points $z = \xi$, η and residues $+1$, -1 respectively. Clebsch and Gordan (1866) devoted Chs. 6–8 in their book to detail properties of this object and regular procedure of derivation the Θ-representation for it (see also Baker 1897). We have from there (Clebsch & Gordan 1866: §§54, 57; Baker 1897: §§171, 187–188)

$$\widetilde{\Pi}_{\alpha,\nu}^{x,z} = \widetilde{\Pi}_{\alpha,\nu}(x) - \widetilde{\Pi}_{\alpha,\nu}(z) = \ln\!\left[\frac{\Theta(\widetilde{\omega}^{x,\alpha} + \boldsymbol{r})}{\Theta(\widetilde{\omega}^{x,\nu} + \boldsymbol{r})} \middle/ \frac{\Theta(\widetilde{\omega}^{z,\alpha} + \boldsymbol{r})}{\Theta(\widetilde{\omega}^{z,\nu} + \boldsymbol{r})}\right],$$

where vector $\boldsymbol{r} = \widetilde{\omega}^{m_g,m} - \widetilde{\omega}^{z_1,m_1} - \cdots - \widetilde{\omega}^{z_{g-1},m_{g-1}}$ is a zero of the Θ-function. In our situation we have $\alpha = \lambda$, $\nu = \infty$, $x = \gamma_k$, $z = \alpha_k$. The representations for zeroes $\boldsymbol{r}$ and the Riemann constants $\boldsymbol{\mathcal{K}}$ are not unique because they depend on a lower bound of the holomorphic integrals $\widetilde{\boldsymbol{\omega}}(z)$. On the other hand the point $\boldsymbol{\eta}$

on the Jacobian contains free constants C_j in (7.4), so that we may simplify the considerations putting m's equal to γ_j's apart from γ_k and set

$$r_j = \widetilde{\omega}_j(\mathcal{P}_1) + \cdots + \widetilde{\omega}_j(\mathcal{P}_{g-1}) + \mathcal{K}_j$$

with arbitrary points $\mathcal{P}_j$, say γ's. We thus arrived at Riemann's function $\Theta\big(\widetilde{\omega}(\mathcal{P}) - e\big)$. As a consequence of these arguments and explicit zeroes of the Ψ in (3.1) we get the following formula (see also Baker (1928): pp. 588–589)

$$\widetilde{\Pi}_{\lambda,\,\infty}^{\gamma_1,\alpha_1} + \cdots + \widetilde{\Pi}_{\lambda,\,\infty}^{\gamma_g,\alpha_g} = \ln \frac{\Theta\big(\widetilde{\omega}(\gamma_1) + \cdots + \widetilde{\omega}(\gamma_g) - \widetilde{\omega}(\lambda) + \mathcal{K}\big)}{\Theta\big(\widetilde{\omega}(\gamma_1) + \cdots + \widetilde{\omega}(\gamma_g) - \widetilde{\omega}(\infty) + \mathcal{K}\big)} + c(x;\tau).$$

The integrals in (7.1) are elementary but not normal. Hence we have an identity

$$\frac{1}{2}\sum_{k=1}^{g} \int_{\alpha_k}^{\gamma_k(x)} \frac{w + \mu(\tau)}{(z - \lambda(\tau))\,w}\,\mathrm{d}z = \widetilde{\Pi}_{\lambda(\tau),\,\infty}^{\gamma_1(x),\alpha_1} + \cdots + \widetilde{\Pi}_{\lambda(\tau),\,\infty}^{\gamma_g(x),\alpha_g} + \sum_{j,k=1}^{g} h_{jk}(\tau)\,\widetilde{\omega}_j\big(\gamma_k(x)\big)$$

with some normalizing constants $h_{jk}(\tau)$. Taking into account Jacobi's problem (7.4) we arrive at the intermediate answer

$$\Psi(x;\tau) = \frac{\Theta\big(\widetilde{\omega}(\lambda(\tau)) + x\boldsymbol{U} - \boldsymbol{C} - \mathcal{K}\big)}{\Theta\big(\widetilde{\omega}(\infty) + x\boldsymbol{U} - \boldsymbol{C} - \mathcal{K}\big)} \exp \sum_{j,k=1}^{g} h_{jk}(\tau)\,\widetilde{\omega}_j\big(\gamma_k(x)\big). \qquad (7.5)$$

Meromorphic part of the Ψ-function has been determined. In order to determine the function $f(x;\tau) = \sum h_{jk}(\tau)\,\widetilde{\omega}_j\big(\gamma_k(x)\big)$ in (7.5) we involve the transformation properties of the Ψ as the function (7.1) on the curve $\mathfrak{S}$. Let $\widehat{a}_k\tau$ and $\widehat{b}_k\tau$ denote linear-fractional transformations of the variable τ corresponding to the cycles a_k and b_k. From the primary representation (3.1) we conclude that $\Psi(x;\tau)$ saves its own form (3.1), up to a multiplier $M(\tau)$, when τ undergoes the transformations $\widehat{\boldsymbol{a}}$, $\widehat{\boldsymbol{b}}$. Hence the formula (7.5) must hold this property. Clearly, the function $f(x;\tau)$ is an entire function of x since $\widetilde{\omega}(\gamma)$'s are everywhere finite. Further, the $\widehat{b}_k$-transformations of the Θ's in (7.5) say that the function f has to be a linear function in x to compensate an exponential multiplier $c(\tau)\exp(-2\pi\mathrm{i}\,x\,U_k)$ in (7.5):

$$f(x;\tau) = \varkappa(\tau)\,x + \mathrm{const}(\tau)\,.$$

Invariance of the Θ-functions in (7.5) with respect to $\widehat{\boldsymbol{a}}$-transformations implies

$$1):\qquad \varkappa(\widehat{a}_k\,\tau) = \varkappa(\tau)\,.$$

The transformations $\widehat{\boldsymbol{b}}$ imply

$$2):\qquad \varkappa(\widehat{b}_k\,\tau) = \varkappa(\tau) + 2\pi\mathrm{i}\,U_k\,.$$

Let τ approaches the pole τ_∞ of the meromorphic function $\lambda(\tau)$. Such a pole is only one and, designating $\xi \equiv \tau - \tau_\infty$, we have

$$\lambda(\tau) = \frac{A^2}{\xi^2} + \frac{B}{\xi} + C + \cdots, \quad \pm\mu(\tau) = \left(\frac{A}{\xi}\right)^{2g+1} + \frac{2\,g+1}{2}\,\frac{B}{A}\left(\frac{A}{\xi}\right)^{2g} + \cdots\,.$$

From these formulas and (3.1) we obtain (extracting terms independent of γ's)

$$\ln\Psi(x;\tau) = \frac{1}{2}\sum_{k=1}^{g}\ln\left\{\frac{A}{\xi^2} + \frac{B}{\xi} + (C - \gamma_k) + \cdots\right\} + \int^{x}\frac{\pm\mu\,\mathrm{d}x}{(\lambda - \gamma_1)\cdots(\lambda - \gamma_g)} =$$

$$= \mathrm{const}(\tau) - \left\{\frac{\sum\gamma_k}{2A^2}\,\xi^2 + \cdots\right\} \pm \int^{x}\left\{\frac{\sum\gamma_k}{A}\,\xi + \cdots\right\}\mathrm{d}x \pm \left\{\frac{A}{\xi} + \frac{B}{2A} + \cdots\right\}x$$

and therefore we get one more property

$$3)\,: \qquad \pm\varkappa(\tau) = \frac{A}{\tau - \tau_\infty} + \frac{B}{2A} + \cdots. \tag{7.6}$$

A function with the properties 1–3) does exist on $\mathfrak{S}$. This is the normal elementary Abelian integral of the second kind $\pm\varkappa(\tau) = \widetilde{\Omega}(\tau)$ with the first order pole at the point τ_∞. Its principal part and a constant in (7.6) are well defined so that this function is completely determined and unique. In turn, the three parameters $\{\tau_\infty, A, B\}$ may be freely chosen, say, $\{0, 1, 0\}$ respectively. The vector $\boldsymbol{U}$ in (7.4–5) becomes the vector of $\boldsymbol{b}$-periods of the integral $\widetilde{\Omega}(\tau)$. Summarizing the arguments above and recovering a normalizing constant

$$\Psi\big(x;\lambda(\tau)\big) = \frac{\Theta\big(\widetilde{\boldsymbol{\omega}}(\infty) - \boldsymbol{D}\big)}{\Theta\big(\widetilde{\boldsymbol{\omega}}(\lambda(\tau)) - \boldsymbol{D}\big)}\,\frac{\Theta\big(\widetilde{\boldsymbol{\omega}}(\lambda(\tau)) + x\boldsymbol{U} - \boldsymbol{D}\big)}{\Theta\big(\widetilde{\boldsymbol{\omega}}(\infty) + x\boldsymbol{U} - \boldsymbol{D}\big)}\,\mathrm{e}^{\widetilde{\Omega}(\tau)\,x} \tag{7.7}$$

we get the spectral properties of the Ψ and complete the proof. $\qquad\square$

From the last expansion of the $\ln\Psi(x;\tau)$ we obtain the formula (2.1). Indeed,

$$\int^{x}\sum_{k=1}^{g}\gamma_k\,\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}\xi}\left\{\ln\Psi(x;\tau) - \widetilde{\Omega}(\tau)\,x\right\}\bigg|_{\xi=0} = \frac{\mathrm{d}}{\mathrm{d}\xi}\ln\frac{\Theta\big(\widetilde{\boldsymbol{\omega}}(\lambda(\tau)) + x\boldsymbol{U} - \boldsymbol{D}\big)}{\Theta\big(\widetilde{\boldsymbol{\omega}}(\lambda(\tau)) - \boldsymbol{D}\big)}\bigg|_{\xi=0}$$

hence, by virtue of the property $\boldsymbol{U} = -\dfrac{\mathrm{d}\widetilde{\boldsymbol{\omega}}(\tau)}{\mathrm{d}\tau}\bigg|_{\tau=\tau_\infty}$, the derivative $\dfrac{\mathrm{d}}{\mathrm{d}\xi}\bigg|_{\xi=0}$ may be replaced by $-\mathrm{d}/\mathrm{d}x$ and we get the final formula

$$u = -2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\ln\Theta\big(x\boldsymbol{U} + \widetilde{\boldsymbol{\omega}}(\infty) - \boldsymbol{D}\big) - \sum_{k=1}^{2g+1}E_k. \tag{7.8}$$

We should conclude here that the spectral and quadrature considerations are mutually replaceable. The explicit transition between these approaches there provides the known Weierstrass's theorem on a permutation of arguments and parameters in the normal Abelian integrals of 3-rd kind

$$\widetilde{\Pi}_{\nu,\mu}(z) - \widetilde{\Pi}_{\nu,\mu}(x) = \widetilde{\Pi}_{z,x}(\nu) - \widetilde{\Pi}_{z,x}(\mu).$$

This fact immediately leads to the equivalence of the 'spectral' formula for the Ψ (Its & Matveev 1975: formulas (4.12–13)) and the quadrature one (4.11) or (7.1). The above mentioned theorem, in our notation, has the form

$$\frac{1}{2}\int_{\alpha_k}^{\gamma_k(x)}\frac{w + \mu}{(z - \lambda)}\frac{\mathrm{d}z}{w} = \frac{1}{2}\int_{\infty}^{\lambda}\left\{\frac{w + \mu_k(x)}{(z - \gamma_k(x))} - \frac{w + \beta_k}{(z - \alpha_k)}\right\}\frac{\mathrm{d}z}{w} + \text{holomorphic part}(x),$$

where $\beta_k^2 = (\alpha_k - E_1) \cdots (\alpha_k - E_{2g+1})$. Since the Ψ contains all the information, we can obtain suitable expressions for all objects of the theory. In particular, Θ-functional representation for 'finite-gap function' of Ermakov (3.4) having numerous applications. Renormalizing it by the formula $\Xi^2 = \Psi_+(x; \lambda) \, \Psi_-(x; \lambda)$ we obtain

$$\Xi^2(x; \lambda) = \text{const.} \; \frac{\Theta\big(\widetilde{\boldsymbol{\omega}}(\lambda) + x\boldsymbol{U} - \boldsymbol{D}\big) \, \Theta\big(\widetilde{\boldsymbol{\omega}}(\lambda) - x\boldsymbol{U} + \boldsymbol{D}\big)}{\Theta^2\big(\widetilde{\boldsymbol{\omega}}(\infty) + x\boldsymbol{U} - \boldsymbol{D}\big)}.$$

'Finite-gap' means that the constant μ in Ermakov's equation (3.4) is not independent of λ: $\mu = \mu(\lambda)$. Note that the equation (3.3) itself exemplifies the integrable and factorizable linear operator pencil with Abelian coefficients (like soliton spectral problems) but its solution, as a counterexample, is not a function of Baker–Akhiezer.

Many of constructions in §§ 3, 5–7 can be carried over to arbitrary spectral problems although not so simply as in the case of the Schrödinger/KdV equation. Nontrivial examples of the Ψ were obtained in (Ustinov 2002) and modifications of Dubrovin's equations and trace formulas in (Brezhnev 2002).

8. Conclusive comments and bibliographical remarks

Apparently Baker (1928: p. 587) was the first to realize the exponential property of the fundamental T-function in disguise, however explicit meromorphic integrals like $\widetilde{\Omega}(\tau)$ are, to all appearances, the result of the modern theory and became a universal property of all the integrable models (Krichever 1970's). Note, that throughout the paper we made no restrictions on the curve (4.6). It may be singular and the theory can be rewritten (with minor changes in § 7) because the degenerated holomorphic integrals $\widetilde{\boldsymbol{\omega}}$, in this case, turn into the integrals $\widetilde{\Pi}$ of the 3-rd kind. Clebsch & Gordan (1866: §43) call the corresponding problem 'extended inversion problem' which is solved by theta as well. Such cases of degenerations were considered also by Baker (1897) and even by Abel (see his Œuvres 1881, **I**: p. 170–...).

(a) On the Θ-series

It is to be noticed, that contrary to commonly accepted (?) viewpoint, the general Θ-series (B 1), as well as its argument $\boldsymbol{z} = \widetilde{\boldsymbol{\omega}}(\mathcal{P}) - \boldsymbol{e}$, ought to be considered not as a special function or formal generalization of Jacobi's θ-function but *regularly* derivable fundamental object for explicit representation of all Abelian integrals, meromorphic/uniformizing functions, exponential Baker–Akhiezer-functions, and the theory as a whole†. Main ideology belongs to K. Weierstrass (1902: pp. 513–538) and was expounded in the books by Tikhomandritskii (1885; 1895: pp. 199–232) before publication of Weierstrassian lectures on Abelian transcendents, but with explicit use of the Riemann surface. Primarily and naturally arising non-canonical form of the Θ-series is derived in these works with basic properties and identities. The canonical formula (B 1), is obtained after the normalization (7.3). See also the book by Clebsch & Gordan (1866: pp. 193–198). For lack of space, we do not pursue these important points here. This will be written up elsewhere. By the same reason we have somewhat reduced exposition between formulas (7.4) and (7.5).

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† Jacobi's case $g = 1$ is not exception (Tichomandritzky, M. 1884 *Math. Ann.* **XXV**, 197–202). Riemann and Baker (1897) do not elucidate an origin of appearance the Θ-series. Hyperelliptic case, based on Weierstrassian lectures, was considered in dissertation by Tikhomandritskii (1885).

(b) *Algebraically integrable Hamiltonian systems*

In what concerns an inverse transition in the thesis on p. 11, i.e. transition from an algebraically integrable Hamiltonian system to some spectral problem, the answer is positive though somewhat noneffective. We assume that separability variables $\{\gamma, \mu\}$ sitting on a curve, say (4.9), exist. We can construct the Baker–Akhiezer function (7.7) and the Schrödinger operator (it certainly exists) for it. The computational part of the Liouville theorem is based on the Hamilton–Jacobi theory of canonical transformations. The variables γ's are always taken as poles of the Ψ:

$$\gamma_k(x): \qquad \Theta\big(\widetilde{\boldsymbol{\omega}}(\infty) + x\boldsymbol{U} - \boldsymbol{D}\big) = 0$$

since they solve the problem (4.10), (7.4). Ostrogradskii's variables $\{u, u_x, \ldots\}$ are constructed explicitly by the trace formulas. Isospectral t-deformations of finite-gap potentials are readily included in (7.2): the quantities d_k become linear functions of $x\boldsymbol{U} + t\boldsymbol{V}$. Accordingly, there is not essential difference between x- and t-equations for the Ψ. Both of these equations (Lax' pair) may be thought of as quadrature integrable (factorizable) spectral problems/pencils: Ermakov–Drach's constant μ becomes an eigenvalue of the second commuting operator connected through the curve with the first one:

$$\widehat{\boldsymbol{A}}([u]; \partial_x)\Psi(x; \lambda) = \mu(\lambda)\Psi(x; \lambda).$$

There are infinitely many Abelian functions built from $\{\gamma, \mu\}$ (followed by physical coordinates $\{p(\gamma, \mu),\, q(\gamma, \mu)\}$) and expressible through the Θ's. Non-constructiveness can appear because transformations between various dynamical systems are not necessary to be canonical. However canonicity is not a necessary attribute of quadrature integrability. *One* fundamental Abelian function determines all the transformations—the potential u.

Appendix A. Note on a transformation of Riccati's equation

R. Rawson. *The Messenger of Mathematics* **XII** (1883), 34–36.

Riccati's equation

$$\frac{dy_1}{dx_1} + a\, y_1{}^2 = \phi(x_1) \dotfill (1)$$

is readily transformed into

$$\frac{dy}{dx} + a\, y^2 = \frac{a_1}{a}\, \phi(P) \left(\frac{dP}{dx}\right)^2 + \frac{3}{4\,a} \left(\frac{\frac{d^2P}{dx^2}}{\frac{dP}{dx}}\right)^2 - \frac{1}{2\,a} \frac{\frac{d^3P}{dx^3}}{\frac{dP}{dx}} \dotfill (2)$$

by means of the two equations

$$x_1 = P, \quad \text{a function of } x \dotfill (3),$$

$$2\,a_1 \left(\frac{dP}{dx}\right)^2 = 2\,a\,y\,\frac{dP}{dx} + \frac{d^2P}{dx^2} \dotfill (4).$$

If, therefore, either (1) or (2) can be integrated, then the other can be integrated also by means of (3) and (4).

An interesting case of the above transformation is when

$$\phi(x_1) = b\,{x_1}^n + \frac{c}{{x_1}^2} \dotfill (5),$$

$$P = x^m \dotfill (6).$$

We then have

$$\frac{dy_1}{dx_1} + a_1{y_1}^2 = b\,{x_1}^n + \frac{c}{{x_1}^2} \dotfill (7),$$

$$\frac{dy}{dx} + a\,y^2 = \frac{a_1 b\,m^2}{a}\,x^{(n+2)m-2} + \frac{(4\,a_1 c + 1)\,m^2 - 1}{4\,a\,x^2} \dotfill (8),$$

$$x_1 = x^m,$$

$$2\,a_1 m\,x^{m-1}\,y_1 = a\,y + \frac{m-1}{2\,x}.$$

In equation (7) let $c = 0$, and $n = -\dfrac{4\,p}{2\,p \pm 1}$, where p is an integer, then (7) and (8) become

$$\frac{dy_1}{dx_1} + a_1{y_1}^2 = b\,x_1^{-\frac{4p}{2p\pm1}} \dotfill (9),$$

$$\frac{dy}{dx} + a\,y^2 = \frac{a_1 b\,m^2}{a}\,x^{\frac{\pm2m}{2p\pm1}-2} + \frac{m^2-1}{4\,a\,x^2} \dotfill (10);$$

and therefore since (9) is soluble, so also is (10).

Equation (10) is made linear by putting

$$y = \frac{1}{a}\frac{dz}{z\,dx},$$

and we thus find

$$\frac{d^2z}{dx^2} = \left\{ a_1 b\,m^2\,x^{\frac{\pm2m}{2p\pm1}-2} + \frac{m^2-1}{4\,x^2} \right\} z,$$

or

$$\frac{d^2z}{dx^2} = \left\{ a_1 b\,(2\,q+1)^2\,x^{\frac{\pm2(2q+1)}{2p\pm1}-2} + \frac{q\,(q+1)}{x^2} \right\} z \dotfill (11),$$

where

$$m = 2\,q + 1.$$

The equations of transformation then become

$$x_1 = x^{2q+1},$$

$$2\,a_1(2\,q+1)\,x^{2q}\,y_1 = a\,y + \frac{q}{x}.$$

The differential equation (11) is of some interest as it includes as a particular case the well-known differential equation

$$\frac{d^2z}{dx^2} = \left\{ a^2 + \frac{q\,(q+1)}{x^2} \right\} z.$$

<div align="right">ROBERT RAWSON.</div>

Appendix B. Notation to §7

The Θ-function corresponds to the symmetrical matrix B_{jk} (7.3)

$$\Theta(\boldsymbol{z}|B) \equiv \Theta(z_1, \ldots, z_g|B) = \sum_{\boldsymbol{N} \in \mathbb{Z}^g} \mathrm{e}^{\pi \mathrm{i}\,\langle B\boldsymbol{N}, \boldsymbol{N}\rangle + 2\pi \mathrm{i}\,\langle \boldsymbol{N}, \boldsymbol{z}\rangle}, \qquad (\mathrm{B}\,1)$$

where $\langle B\boldsymbol{N}, \boldsymbol{N}\rangle = \sum B_{jk}N_jN_k$ and $\langle \boldsymbol{N}, \boldsymbol{z}\rangle = N_1 z_1 + \cdots + N_g z_g$. Canonical base of cycles (a_j, b_j) on $\mathfrak{S}$ is chosen according to the intersection scheme: $a_j \circ b_k = \delta_{jk}$. The normal Abelian integrals of the 2, 3-rd kind and their periods have the form:

$$\widetilde{\Omega} = \frac{1}{\tau - \tau_\infty} + c_1(\tau - \tau_\infty) + \cdots, \qquad \oint_{a_k} \mathrm{d}\widetilde{\Omega} = 0, \qquad \oint_{b_k} \mathrm{d}\widetilde{\Omega} = 2\,\pi\,\mathrm{i}\,U_k,$$

$$\widetilde{\Pi}_{\alpha,\nu}(\tau) = \ln \frac{\tau - \tau_\alpha}{\tau - \tau_\nu} + \cdots, \qquad \oint_{a_k} \mathrm{d}\widetilde{\Pi}_{\alpha,\nu} = 0, \qquad \oint_{b_k} \mathrm{d}\widetilde{\Pi}_{\alpha,\nu} = 2\,\pi\,\mathrm{i}\left(\widetilde{\omega}_k(\nu) - \widetilde{\omega}_k(\alpha)\right).$$

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